

Quatum-like model of processing of information in the brain based on classical electromagnetic field

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Abstract

We propose a model of quantum-like (QL) processing of mental information. This model is based on quantum information theory. However, in contrast to models of “quantum physical brain” reducing mental activity (at least at the highest level) to quantum physical phenomena in the brain, our model matches well with the basic neuronal paradigm of the cognitive science. QL information processing is based (surprisingly) on classical electromagnetic signals induced by joint activity of neurons. This novel approach to quantum information is based on representation of quantum mechanics as a version of classical signal theory which was recently elaborated by the author. The brain uses the QL representation (QLR) for working with abstract concepts; concrete images are described by classical information theory. Two processes, classical and QL, are performed parallelly. Moreover, information is actively transmitted from one representation to another. A QL concept given in our model by a density operator can generate a variety of concrete images given by temporal realizations of the corresponding (Gaussian) random signal. This signal has the covariance operator coinciding with the density operator encoding the abstract concept under consideration. The presence of various

temporal scales in the brain plays the crucial role in creation of QLR in the brain. Moreover, in our model electromagnetic noise produced by neurons is a source of superstrong QL correlations between processes in different spatial domains in the brain; the binding problem is solved on the QL level, but with the aid of the classical background fluctuations.

1 Introduction

Last years the hypothesis that the brain processes information (at least partially) by using quantum-like (QL) representation of probabilities by complex amplitudes was discussed from various viewpoint by numerous authors, e.g., [1]–[22], see also [12]– [17] for statistical studies confirming (at least preliminary) this hypothesis. In general such processing need not be based on the physical quantum brain (cf., e.g., Homeroff, Penrose, and Vitiello, see [1]–[5]) – quantum physical carriers of information. In our approach the brain created the QL representation (QLR) of information in Hilbert space. It uses quantum information rules in decision making. The existence of such QLR was (at least preliminary) confirmed by experimental data from cognitive psychology and economics.¹ The violation of LTP in these experiments is an important sign of nonclassicality of data – the interference effect. Moreover, recently so called *constructive wave function approach* was developed: data violating LTP can be represented by complex probability amplitudes by application of a special algorithm – QLR-algorithm [24], [22]. Recently there were also developed QL models of decision making, see [13]–[15], [11], [21], [29], .

The next natural step is to try to find possible physical realizations of QLR in the brain. One of possibilities is to appeal to quantum physics of microprocesses in the brain – the quantum brain, see, e.g.,

¹In [12]– [17] we performed experiments with recognition of ambiguous figures which demonstrated interference effects for incompatible recognition tasks in the form of violation of the law of total probability (LTP), see [23], [24], [22] for details. (One can speak about non-Kolmogorovness of probabilistic data, i.e., impossibility to describe it by using the Kolmogorov probability model [25].) In [13]–[15], [22] it was demonstrated that statistical data from well known experiments in cognitive economics, Shafir and Tversky [26], [27], on so called disjunction effect in decision making also exhibit interference effect by violating LTP. We recall that the disjunction effect is related to violation of Savage’s sure thing principle [28] – the basic principle of the modern economic theory, the principle of rationality of decision making.

Homeroff and Penrose, see [1]–[4]. However, (surprisingly) it is possible to proceed even in the classical field framework, i.e., to create a classical wave model producing QLR, see [31]–[33]. Neurophysiologically our model is based on *variety of time scales in the brain*, see e.g. [30]. Each pair of scales (fine – the background fluctuations of electromagnetic field and rough – the mental image scale) induces the QL representation. The background field plays the crucial role in creation of “superstrong QL correlations” in the brain.

We propose a classical (!) wave model which reproduces probabilistic effects of quantum information theory. Why do we appeal to classical electromagnetic fields in the brain and not to quantum phenomena? In neurophysiological and cognitive studies we see numerous classical electromagnetic waves in the brain. Our conjecture is that these waves are carriers of mental information which is processed in the framework of quantum information theory.

In the quantum community there is a general opinion that quantum effects can not be described by classical wave models (however, cf. Schrödinger; we also can mention works of H. F. Hofmann who demonstrated that one can use classical electromagnetic fields to describe a variety of “purely quantum effects” [34]). Even those who agree that the classical and quantum interferences are similar emphasize the role of quantum entanglement and its irreducibility to classical correlations (however, cf. Einstein-Podolsky-Rosen). It is well known that entanglement is crucial in quantum information theory. Although some authors emphasize the role of quantum parallelism in quantum computing, i.e., superposition and interference, experts know well that without entanglement the quantum computer is not able to beat the classical digital computer.

Recently the author proposed a classical wave model reproducing all probabilistic predictions of quantum mechanics, including correlations of entangled systems, so called *prequantum classical statistical field theory* (PCSFT)[31, 32] and see paper[33] for the recent model for composite systems. It seems that, in spite of the mentioned common opinion, the classical wave description of quantum phenomena is still possible.

In this paper we apply PCSFT to model QL processing of information in the brain on the basis of classical electromagnetic fields. This model is based on the presence of various *time scales in the brain*. Roughly speaking each pair of time scales, one of them is fine – the background fluctuations of electromagnetic (classical) field in the

brain, and another is rough – the mental image scale, can be used for creation of QLR in the brain. The background field (background oscillations in the brain) which is an important part of our model plays the crucial role in the creation of “superstrong QL correlations” in the brain, cf. Hofmann [34]. These mental correlations are nonlocal due to the background field. These correlations might provide a solution of the *binding problem*.

Each such a pair of time scales, (fine, rough), induces QLR of information. As a consequence of variety of time-scales in the brain, we get a variety of QL representations serving for various mental functions. This QL model of brain’s functioning was originated in author’s paper[35]. The main improvement of the “old model” is due to a new possibility achieved recently by PCSFT: to represent the quantum correlations for entangled systems as the correlations of the classical random field, so to say prequantum field. This recent development also enlightened the role of the *background field*, vacuum fluctuations. We now transfer this mathematical construction designed for quantum physics to the brain science. Of course, it is a little bit naive model, since we do not know the “QL code” used by the brain: the correspondence between images and probability distributions of random electromagnetic fields in the brain.

We speculate that decision making through nonclassical LTP is based on a *wave representation of information* in the brain. The brain is full of classical electromagnetic radiation. May be the brain was able to create QLR of information via classical electromagnetic signals, cf. K.-H. Fichtner, L. Fichtner, W. Freudenberg and M. Ohya [19]. We also make a remark on the approach of G. Vitiello [5]. It differs essentially from majority of approaches to quantum physical brain. In contrast to others, Vitiello uses not quantum mechanics, but quantum field theory, In some sense his model is closer to PCSFT. He also operate with macroscopic electromagnetic signals, but for him these signals are described by the quantum field model.

It is well known that classical waves produce superposition and, hence, violate LTP. However, quantum information processing is based not only on superposition, but also on ENTANGLEMENT. It is the source of superstrong nonlocal correlations. Correlations are really superstrong – violation of Bell’s inequality. Can entanglement be produced by classical signals? Can quantum information processing be reproduced by using classical waves? Surprisingly, the answer is positive.

The crucial element of our classical wave model of brain's functioning is the presence of the random background field (in physics fluctuations of vacuum, in the cognitive model – background fluctuations of the brain). Such a random background increases essentially correlations between different mental functions, generates nonlocal presentation of information. As was already remarked, we might couple these nonlocal representation of information to the binding problem:

“How the unity of conscious perception is brought about by the distributed activities of the central nervous system.”

2 Why may brain use the quantum-like representation of information based on classical electromagnetic waves?

As we emphasized, the deep neurophysiological studies demonstrated that the brain definitely processes the information by using classical electromagnetic signals. We would like to apply the results of these studies and propose a classical-signal model of the brain functioning. However, we do not plan just to explore the standard classical signal theory. We speculate that information processing in the brain should be described by the mathematical formalism of quantum mechanics and that classical electromagnetic waves are used by the brain to create QLR.

“Why was the brain not satisfied with the classical signal processing?”

What are advantages of the QL processing of information (even with classical field)?”

2.1 Incomplete processing of information

If we speculate that in physics quantum probabilistic behavior can be expressed through classical random waves, we definitely reject Bohr's thesis on completeness of QM. (This Einsteinian attitude is characteristic only for this chapter, in previous chapters we were able to proceed even with the orthodox Copenhagen interpretation.) Here we have to assume that the QM formalism provides only an approximate description of processes in the micro world. In such an approach the

main difference of quantum processing of information from classical is that the first one provides a possibility to ignore consistently a part of information, to make a consistent cutoff of information (described by the mathematical formalism of QM).

Operation through incomplete information processing is very profitable for cognitive systems. An important part of cognition is the extraction of a part of information from huge information flows coming to the brain. By operating in the QL-framework, for example, on the basis of the wave representation, the brain gets a possibility to work harmonically with incomplete information. This is one of sources of creation of the QL- processing in the brain.

By ignoring a part of information the brain is able to create abstract mental images, ideas, concepts, categories. Another advantage is incredible increasing of the speed of computations. Here we speak about computations based on classical electromagnetic signals, but performed on the basis of quantum formalism.

2.2 Background noise: How can the worst enemy become the best friend?

By PCSFT the standard QM formalism provides a possibility to extract signals (in fact, their averages) from the noisy background, see section 8.5. QM can be interpreted as a kind of renormalization theory which is applicable to signals with irreducible noise. In quantum physics this is the noise of vacuum fluctuations, the background field. It seems that this noise is a fundamental feature of space [36], [37]; one cannot hope to isolate a signal from these fluctuations; the only possibility is to take them into account consistently. We now project this situation to brain's functioning. The brain is a complex electric system; its functioning has definitely to induce noise; more complex brains are more noisy; higher brain's activity is also noisier. Of course, in the process of evolution cognitive systems might try reduce the impact of the background noise in the brain. However, it is clear that it would be a very complicated task: it seems impossible to isolate signals in the brain from the background fluctuations (induced by a huge number of neurons).

We speculate that the brain chosen another way to evolve, namely, to elaborate a procedure to extract mental images from signals on the noisy background, i.e., proceeding with the noise, but, finally, having a procedure (consistent for different signals) for extracting images. By

PCSFT processing of information on the basis of the QLR provides such a possibility.

Moreover, in physics (by PCSFT) the background field is a source of superstrong nonlocal correlations between entangled systems – “entangled classical waves” in the PCSFT-framework. By projecting this situation to the brain functioning we see that the brain can get the great advantage from the presence of the background fluctuations. They produce entanglement between processes in different (including spatially separated) domains of the brain, between different mental functions, see section 9.2 for further discussion.

2.3 Joint processing of an abstract image and its concrete realizations

By PCSFT a random electromagnetic field is represented in the QL-way by its covariance operator (=“density operator”). In the cognitive model the QL-representation corresponds to abstract mental images. Thus they are given by covariance operators. To process such QL-images the brain is not interested in the complex structure of random fluctuations of classical signals; the brain operates with images encoded by operators – covariance operators (matrices) of classical random signals. However, in this process the brain might need to proceed from the abstract image to its concrete realization. For example, the brain can operate with abstract notions, e.g., house and tree, but it can switch to the concrete house and the concrete tree. In processing of the first type (the QL-processing) the brain operates with operators D_{house} and D_{tree} and in processing of the second type (the classical processing) the brain has to switch to the classical signals encoding this house and this tree.

We remark that two different realizations of the random signal with the fixed covariance operator can differ essentially, so they really can encode two different houses. Take a coin. Consider a series of its tossing, e.g., a few thousands: $x = (x_1, x_2, \dots, x_n)$, where $x_j = 0, 1$ are labels of coin’s sides. After this tossing was finished, start a new series: $y = (y_1, y_2, \dots, y_n)$. Although both sequences are samples of the same random law, they can differ essentially bit wise. Thus we can encode “house” (as the abstract notion) by the probability law of this coin, but two concrete houses are encoded by sequences x and y .

If the brain works in the QL (abstract) regime, it recognizes only encoding of the corresponding probability laws; if it works in the clas-

sical regime it has to use essentially more processing resources, since it operates not with the codes of probabilistic laws, but with the real data streams.

In the QL-model induced by PCSFT we consider only *Gaussian random fields* with zero averages (symmetric oscillations). Such random fields are uniquely determined by covariance operators. The former remark on encoding by using the coin and its tossings can be modified in the following way. We have a Gaussian random generator producing vector data, each vector (realization) has dimension m . Starting with some input given by the vector x_0 the generator produces the stream of vectors x encoding the concrete house. Starting with another input x'_0 the generator produces another stream of random data y encoding another concrete house. The concept of house is represented by the covariance operator $D = D_{\text{house}}$ of this random generator, the density operator in the QL-formalism.

3 Prequantum classical statistical field theory: noncomposite systems

Quantum mechanics (QM) is a statistical theory. It cannot tell us anything about an individual quantum system, e.g., electron or photon. It predicts only probabilities for results of measurements for ensembles of quantum systems. Classical statistical mechanics (CSM) does the same. Why are QM and CSM based on different probability models?

In CSM averages are given by integrals with respect to probability measures and in QM by traces. In CSM we have:

$$\langle f \rangle_\mu = \int_M f(\phi) d\mu(\phi), \quad (1)$$

where M is the state space. In probabilistic terms: there is given a random vector $\phi(\omega)$ taking values in M . Then $\langle f \rangle_\phi = Ef(\phi(\omega)) = \langle f \rangle_\mu$. In QM the average is given by the operator trace-formula:

$$\langle \hat{A} \rangle_\rho = \text{Tr} \rho \hat{A}. \quad (2)$$

This formal mathematical difference induces the prejudice on fundamental difference between classical and quantum worlds. Our aim is to show that, in spite of the common opinion, quantum averages can be easily represented as classical averages and, moreover, even correlations between entangled systems can be expressed as classical correlations (with respect to fluctuations of classical random fields).

3.1 Einstein's dreams:

Albert Einstein did not believe in irreducible randomness, completeness of QM. He dreamed of a better, so to say “prequantum”, model [38]:

1). **Dream 1.** A mathematical model reducing quantum randomness to classical.

2). **Dream 2.** Renaissance of causal description.

3). **Dream 3.** Instead of particles, classical fields will provide the complete description of reality – reality of fields [38]:

“But the division into matter and field is, after the recognition of the equivalence of mass and energy, something artificial and not clearly defined. Could we not reject the concept of matter and build a pure field physics? What impresses our senses as matter is really a great concentration of energy into a comparatively small space. We could regard matter as the regions in space where the field is extremely strong. In this way a new philosophical background could be created.”

The real trouble of the prequantum wave model (in the spirit of early Schrödinger) are not various NO-GO theorems (e.g., the Bell inequality [24, ?]), but the problem which was recognized already by Schrödinger. In fact, he gave up with his wave quantum mechanics, because of this problem: *A composite quantum system cannot be described by waves on physical space!* Two electrons are described by the wave function on \mathbf{R}^6 and not by two wave on \mathbf{R}^3 .

Einstein also recognized this problem [38]: “For one elementary particle, electron or photon, we have probability waves in a three-dimensional continuum, characterizing the statistical behavior of the system if the experiments are often repeated. But what about the case of not one but two interacting particles, for instance, two electrons, electron and photon, or electron and nucleus? We cannot treat them separately and describe each of them through a probability wave in three dimensions...”

3.2 Quantum system = classical random field

Einstein's Dreams 1 and 3 came true in PCSFT (but not Dream 2!) – a version of CSM in which fields play the role of particles.² In particular,

²It seems surprising that, although Dream 1 came true, Dream 2 cannot. The situation differs essentially from CSM where dynamics of probability distribution given by the Liouville equation can be reduced to the deterministic Hamiltonian dynamics. The main

composite systems can be described by vector random fields, i.e., by the Cartesian product of state spaces of subsystems and not the tensor product. The basic postulate of PCSFT can be formulated in the following way:

A quantum particle is the symbolic representation of a “prequantum” classical field fluctuating on the time scale which is essentially finer than the time scale of measurements.

The prequantum state space $M = L_2(\mathbf{R}^3)$, states are fields $\phi : \mathbf{R}^3 \rightarrow \mathbf{R}$; “electronic field”, “neutronic field”, “photonic field” - classical electromagnetic field. An ensemble of “quantum particles” is represented by an ensemble of classical fields, probability measure μ on $M = L_2(\mathbf{R}^3)$, or random field $\phi(x, \omega)$ taking values in $M = L_2(\mathbf{R}^3)$. For each fixed value of the random parameter $\omega = \omega_0$, $x \rightarrow \phi(x, \omega_0)$ is a classical field on physical space.

3.3 Density operator = covariance operator

Each measure (or random field) has the covariance operator, say D . It describes correlations between various degrees of freedom.

The map $\rho \mapsto D = \rho$ is one-to-one between density operators and the covariance operators of the corresponding prequantum random fields – in the case of noncomposite quantum systems. In the case of composite systems this correspondence is really tricky.

Thus each quantum state (an element of the QM formalism) is represented by the classical random field in PCSFT. The covariance operator of this field is determined by the density operator. We also postulate that the prequantum random field has *zero mean value*. These two conditions determine uniquely Gaussian random fields. We restrict our model to such fields. Thus by PCSFT quantum systems are Gaussian random fields.

Finally, we remind that the covariance operator D of a random field ϕ is defined by its bilinear form ($u, v \in H$) :

$$\langle Du, v \rangle = E \langle \phi, u \rangle \langle v, \phi \rangle = E \left(\int_O \phi(x, \omega) \overline{u(x)} dx \right) \left(\int_O v(x) \overline{\phi(x, \omega)} dx \right) \quad (3)$$

or by using the probability distribution μ of the random field:

$$\langle Du, v \rangle = \int_H \left(\int_O \phi(x) \overline{u(x)} dx \right) \left(\int_O v(x) \overline{\phi(x)} dx \right) d\mu(\phi). \quad (4)$$

difference is due to the presence of the background fluctuations – irreducible noise.

3.4 Quantum observable = quadratic form

The map $\widehat{A} \rightarrow f_A(\phi) = (\widehat{A}\phi, \phi)$ establishes one-to-one correspondence between quantum observables (self-adjoint operators) and classical physical variables (quadratic functionals of the prequantum field).

It is easy to prove that following equality holds:

$$E f_A(\phi(\omega)) = \int_M f_A(\phi) d\mu(\phi) = \text{Tr} \rho \widehat{A}. \quad (5)$$

In particular, for a pure quantum state ψ , consider the Gaussian measure with zero mean value and the covariance operator $\rho = \psi \otimes \psi$ (the orthogonal projector on the vector ψ), then

$$\int_M f_A(\phi) d\mu(\phi) = (\widehat{A}\psi, \psi).$$

This mathematical formula coupling integral of a quadratic form and the corresponding trace is well known in measure theory. Our main contribution is coupling of this mathematical formula with quantum physics.

This is the end of the story for quantum noncomposite systems, e.g., a single electron or photon[31, 32].

3.5 Beyond QM

In fact, PCSFT not only reproduces quantum averages, but it also provides a possibility to go beyond QM. Suppose that not all prequantum physical variables are given by QUADRATIC forms, consider more general model, all smooth functionals $f(\phi)$ of classical fields. We only have the illusion of representation of all quantum observables by self-adjoint operators.

The map

$$f \mapsto \widehat{A} = f''(0)/2 \quad (6)$$

projects smooth functionals of the prequantum field (physical variables in PCSFT) on self-adjoint operators (quantum observables). Then quantum and classical (prequantum) averages do not coincide precisely, but only approximately:

$$\int_M f_A(\phi) d\mu(\phi) = \text{Tr} \rho \widehat{A} + O(\tau/T), \quad (7)$$

where T is the time scale of measurements and τ the time scale of fluctuations of prequantum field. The main problem is that PCSFT

does not provide a quantitative estimate of the time scale of fluctuations of the prequantum field. If this scale is too fine, e.g., the Planck scale, then QM is “too good approximation of PCSFT”, i.e., it would be really impossible to distinguish them experimentally. However, even a possibility to represent QM as the classical wave mechanics can have important theoretical and practical applications. And in the present paper we shall use the mathematical formalism of PCSFT to model brain’s functioning. Although even in this case the choice of the scale of fluctuations is a complicated problem, we know that it is not extremely fine; so the model can be experimentally verified (in contrast to Roger Penrose we are not looking for cognition at the Planck scale!).

4 Cognitive model: Two regimes of brain’s functioning

We now turn to considerations of section 2.3 and proceed on the basis of the short presentation of PCSFT given in section 3. At the moment we consider one fixed mental function of the brain, say F , which is physically concentrated in some spatial domain $O \subset \mathbf{R}^3$ of the brain. We shall come to the model of QL cooperation of a few mental functions after the presentation of PCSFT for composite systems, section 8 (“entanglement of mental functions”).

4.0.1 Classical regime

By getting an input ϕ_0 (from environment or another mental function) the mental function F produces a random signal $\phi(x, \omega)$ – a classical electromagnetic field resulting from neuronal activity.³ It is a random signal depending on the chance parameter ω . For each ω_0 , this electromagnetic field, $x \rightarrow \phi(x, \omega_0)$, is distributed on the domain O .

We use the complex representation for the electromagnetic field, the Riemann-Silberstein representation:

$$\phi(x) = E(x) + iB(x),$$

³Thus we consider the ensemble of neurons which are located in O . By our model the brain does not so much interested in the “private life” of individual neurons, i.e., the frequency of spikes and so on, cf. [39]. It is only interested in the electromagnetic field induced by activity of these neurons.

where $E(x) = (E_1(x), E_2(x), E_3(x))$ and $B(x) = (B_1(x), B_2(x), B_3(x))$ are the electric and magnetic components, respectively.

In our model *each concrete mental image is associated with a random signal*. Its mental features, “*qualia*”, are given by functionals of this signal. In the simplest case these are quadratic forms of the signal.

The main problem is to create the classical signal code, i.e., to establish correspondence between random signals and mental images as well as between field-functionals and qualia of images. We speculate that at least some field-functionals represent *emotions* related to the mental image (which is represented by the classical electromagnetic signal). Consider a number of emotions, say $\mathcal{E}_1, \dots, \mathcal{E}_k$, related to some image, say MI_ϕ (associated with the signal ϕ). Then the mental function F physically operates with the corresponding field-functionals; in the simplest case these are quadratic functionals and they can be represented by integral kernels:

$$f_{\mathcal{E}_j}(\phi) = \int_{O \times O} K_j(x, y) \phi(x) \overline{\phi(y)} dx dy. \quad (8)$$

However, in the classical regime *nonquadratic functionals* are also in the use; e.g.,

$$f(\phi) = f_{\mathcal{E}}(\phi) + \int_{O \times O \times O} K(x, y, z) \phi(x) \phi(y) \overline{\phi(z)} dx dy dz,$$

where $f_{\mathcal{E}}$ is the functional of the form (8).

Remark 1. (Spatial distribution of qualia) By considering integral functionals of the classical electromagnetic field we suppose that the F -function performs integration of a signal over its domain of spatial concentration. Thus we consider only spatially concentrated mental functions. If a mental function F is concentrated in a domain $O = \cup_k O_k$, where O_k are located in the brain far away from each other, then we represent F as collection of “elementary mental functions” F_k concentrated in domains O_k . Some qualia of F are associated with elementary functions F_k . However, there are also exist global qualia which are obtained by summation of local ones (so integration on each O_k and then collection and summation in a special center).

Since signals are random, field functionals are fluctuating quantities – random variables: $\xi_\phi(\omega) = f(\phi(\omega))$. It is clear that the brain

cannot operate with such unstable mental entities. Thus it has to produce averages of emotions and operate with them. It will be especially clear in the time-representation of random signals, see section 5. Let μ be the probability distribution of a random signal ϕ . Then emotions are quantified by averages:

$$\langle f_{\mathcal{E}_j} \rangle = \int_H f_{\mathcal{E}_j}(\phi) d\mu(\phi), \quad (9)$$

where $H = L_2(O)$. In our model not only emotions, but all qualia are quantified by averages.

It is clear that quantification of each qualia consumes brain's resources. Therefore only a special class of qualia (in particular, emotions) is associated with each mental image. How does the mental function F select them is the open question. The crucial point is that in principle any two emotions \mathcal{E}_1 and \mathcal{E}_2 or other qualia can be associated with the image MI_ϕ and quantified. This *total compatibility of emotions and qualia in general* may induce some problems. For example, it is not always profitable for survival to combine some emotions. We shall see that the situation is totally different in QL processing of information.

Functionals of classical electromagnetic signals represents not only emotions, but even other qualia of the image MI_ϕ . For example, in PCSFT we have the energy variable (representing the intensity of a signal):

$$f_I(\phi) = \int_O |\phi(x)|^2 dx = \int_O (E^2(x) + B^2(x)) dx. \quad (10)$$

We relate this functional to the *intensity of feeling* of the image MI_ϕ . This intensity is quantified as

$$\begin{aligned} \langle f_I \rangle &= \int_H f_I(\phi) d\mu(\phi) = \int_H \left(\int_O |\phi(x)|^2 dx \right) d\mu(\phi), \quad (11) \\ &= \int_H \left(\int_O (E^2(x) + B^2(x)) dx \right) d\mu(\phi). \end{aligned}$$

In QM the position observable is given by the multiplication operator

$$\hat{x}\phi(x) = x\phi(x)$$

and in PCSFT it is represented by the field functional:

$$f_x(\phi) = (\hat{x}\phi, \phi) = \int_O x|\phi(x)|^2 dx = \int_O x(E^2(x) + B^2(x)) dx. \quad (12)$$

What quale can be coupled to this functional?

Already here, on the level of classical mental processing, mathematics lead us to the notion of *conjugate qualia of a mental image*. For example, consider the momentum functional in PCSFT:

$$f_p(\phi) = (\widehat{p}\phi, \phi) = \int_{\mathbf{R}^3} p|\tilde{\phi}(p)|^2 dp, \quad (13)$$

where $\tilde{\phi}(p)$ is the Fourier transform of the signal $\phi(x)$. What is a cognitive interpretation of conjugation between qualia given by functionals f_x and f_p ?

It seems that in the classical regime the brain can process conjugate qualia simultaneously. In the case of “position and momentum” functionals it is simultaneous processing in the spatial and frequency representations.

Classical mental coding: How does the brain associate the mental image MI_ϕ with a classical signal ϕ ? In our model it is done through calculation of its covariance operator $D = D(\phi)$. This association, mental image – covariance operator, is especially natural in the time representation of random signals, see section 5.

4.0.2 Quantum-like regime

We are now interested in the QLR (quantum-like representation) of information. In QLR the brain operates with density operators which represent not only concrete mental images (coming from the classical regime of mental processing), but also abstract concepts (of different levels of abstraction, see section 7 for details) which do not correspond to classically produced images.

In QLR the brain’s state space is space of density operators $\mathcal{D}(H)$, $H = L_2(O)$. In principle each density operator can be used as a QL state of the brain. However, it is natural to assume that each mental function F operates in its own subspace $\mathcal{D}_F(H)$ of $\mathcal{D}(H)$.

In the standard QM a system has not only the state, but also “properties” or (depending on interpretation) there are defined observables on this system (in this state) – e.g., the energy observable, the coordinate observable and so on. By QM they are represented by self-adjoint operators.

To simplify mathematics we shall consider only bounded (continuous) operators; denote the space of all bounded self-adjoint operators by the symbol $\mathcal{L}_s(H)$. For a given quantum state $\rho \in \mathcal{D}(H)$ and

observable $\hat{A} \in \mathcal{L}_s(H)$, the QM formalism gives the average of this observable in this state, see (2).

We encode qualia of a QL cognitive image MI_ρ (which is encoded by a density operator ρ) by self-adjoint operators; thus qualia of a mental image (in QLR of information) are described by the space $\mathcal{L}_s(H)$. They are quantified by their averages, via (2).

Opposite to classical processing, in QLR the brain cannot quantify all qualia simultaneously. There exist *incompatible qualia*; in particular, incompatible emotions. The QL brain can select different representations of the mental image MI_ρ and different collections of compatible qualia of the image. The QL brain escapes the simultaneous use of e.g. some emotions (“incompatible emotions”). In this way QLR-processing differs essentially from classical processing. We can speculate that in the process of evolution the brain created (on the basis of experience) commutative algebras corresponding to compatible qualia.

4.0.3 Coupling between classical and quantum-like representations

The crucial point of QLR of information is that this “operator-thinking” is naturally coupled with processing of classical electromagnetic signals. On the level of mental images we have:

aMI). From the classical regime to QL: a classical signal ϕ induces the mental image MI_ϕ given by its covariance operator $D = D(\phi)$ and it is transferred to QLR through normalization by the trace: $D \rightarrow \rho = D/\text{Tr}D$. Thus there is a map from classical mental images to QL mental images: $MI_\phi \rightarrow MI_\rho$.

bMI). From QL to classical: a QL mental image MI_ρ can be represented by a classical (Gaussian) signal $\phi = \phi_\rho$ with the covariance operator ρ , i.e., by the image MI_ϕ .

On the level of qualia we have:

aQU). From classical regime to QL: each functional of classical field, $f(\phi)$, is represented by its second derivative – self-adjoint operator, see (6).

bQU). From QL to classical: each quantum quale (given by a self-adjoint operator) is represented by its quadratic form.

Since aQU is not one-to-one, i.e., since a huge class of different classical qualia given by various functionals of the electromagnetic

field (all functionals with the same second derivative) is mapped into the same self-adjoint operator, QLR makes the mental picture less rich than it was in the classical representation. The same can be said about aMI. Various classical signals, concrete mental images, can have the same covariance operator (we do not claim that input signals are obligatory Gaussian, so there is no one-to-one correspondence), cf. with a general discussion on mental prespace [?].

Take a classical mental image MI_ϕ . It is represented by the covariance operator $D = D(\phi)$. Of course, it can be mapped to a QL image MI_ρ , see aIM. Values of all QL qualia can be obtained by scaling from values of corresponding classical qualia, since, for any operator $\widehat{A} \in \mathcal{L}_s(H)$,

$$\begin{aligned} \langle \widehat{A} \rangle_\rho &= \text{Tr} \rho \widehat{A} = \frac{1}{\text{Tr} D} \text{Tr} D \widehat{A} \\ &= \frac{1}{\text{Tr} D} \int_H f_A(\phi) d\mu(\phi) = \frac{1}{\text{Tr} D} \langle f_A \rangle_\phi, \end{aligned}$$

where μ is the probability distribution of the signal. We remark that

$$\text{Tr} D = \int_H \left(\int_O |\phi(x)|^2 dx \right) d\mu(\phi) = \langle f_I \rangle$$

is the intensity of the signal or in the cognitive model the intensity of feeling of the mental image MI_ϕ . Thus QL qualia are normalized by the intensity of feeling:

$$\langle \widehat{A} \rangle_\rho = \frac{\int_H f_A(\phi) d\mu(\phi)}{\int_H \left(\int_O |\phi(x)|^2 dx \right) d\mu(\phi)}.$$

QL processing of all mental images is performed on the same level of intensity of feeling, so it is “*calm thinking*”.

We remark once again that some classical qualia do not have quantum counterpart.

5 Classical regime: Time representation

As usual in signal theory, we can switch from the ensemble representation for averages to the time representation (under the standard assumption of *ergodicity*). Thus, instead of a random field $\phi(x, \omega)$,

which is distributed with some probability distribution $d\mu(\phi)$ on H , we consider a time dependent signal

$$\phi(s) \equiv \phi(s, x),$$

where $x \in O, s \in [0, +\infty)$. Then, for each functional $f(\phi)$ such that $\int_H |f(\phi)| d\mu(\phi) < \infty$, we have (by ergodicity):

$$\langle f \rangle_\mu \equiv \int_H f(\phi) d\mu(\phi) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(\phi(s)) ds \equiv \langle f \rangle_\phi. \quad (14)$$

Consider two time scales: τ is a fine scale and $T \gg \tau$ is a rough time scale. In QM the latter is the scale of measurements and τ is the scale of fluctuations of the prequantum field.⁴ In cognitive science we use the following interpretation of time scales: T is the scale of the QLR and τ is the scale of the real physical processing of the electromagnetic signal in the brain. Thus

$$\langle f \rangle_\phi \approx \frac{1}{T} \int_0^T f(\phi(s)) ds, \quad (15)$$

where s denotes the time variable at the τ -scale. We call the T -scale the *mental time scale*; we can also speak about *psychological time*. The T -scale is the scale of creation of mental images by the brain. The τ -scale is the physical processing scale or *premental time scale*. Neurophysiological theoretical and experimental studies provide the following estimate of relative magnitudes of these time scales. If we select $\tau = 1$ mls., then $T \approx 80$ mls.

For each signal $\phi(s, x), x \in O$, the brain can find its qualia, e.g., the strength of feeling of this image:

$$\langle f \rangle_\mu \approx \frac{1}{T} \int_0^T \left(\int_O (E^2(s, x) + B^2(s, x)) dx \right) ds. \quad (16)$$

In particular, emotions (special qualia) are given by such functionals, e.g., $f_{\text{anger}}, f_{\text{sadness}}, \dots$. Our formal mathematical model cannot provide

⁴PCSFT does not predict the magnitude of the scale of prequantum field fluctuations. One may speculate (motivated in particular by cosmology and string theory), cf. G. 't Hooft [36], [37] that it has to be the Planck scale $\tau_P \approx 10^{-44}$ s. If it were really the case, then prequantum fluctuations have only a theoretical value: they will be never approached experimentally. However, in [40] we discussed a possibility that the prequantum scale may have a larger magnitude and, hence, fluctuations will be soon or later approached experimentally.

the form of concrete emotion-functionals. We hope that in future it will be described as the result of neurophysiological and cognitive studies.

In the time representation the covariance operator (its bilinear form) of the signal $\phi(s, x)$ is given by

$$\begin{aligned} (Du, v) &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \left(\int_O u(x) \overline{\phi(s, x)} dx \int_O \phi(s, x) \overline{v(x)} dx \right) ds, \\ &\approx \frac{1}{T} \int_0^T \left(\int_O u(x) \overline{\phi(s, x)} dx \int_O \phi(s, x) \overline{v(x)} dx \right) ds, \end{aligned} \quad (17)$$

where $u(x), v(x)$ are two “test signals”, $u, v \in L_2(O)$.

6 Classical Signal Processing of Mental Images

This section contains a detailed presentation of the classical processing of information, see section 4.0.1. We proceed in the time representation of random signals.

CSP1. Electromagnetic field basis of mental images. Inputs from external and internal worlds induce electromagnetic signals in the brain.

Each signal has a variety of qualia; in particular, emotions associated with the signal $\phi(s, x)$. Qualia are realized by various functionals, $\phi \mapsto f(\phi)$, of the signal. They are quantified by averages of these functionals, $f \mapsto \langle f \rangle_\phi$, see (14) and (16). In principle all possible qualia (e.g., emotions) can be jointly associated with $\phi(s, x)$.

The physical dynamics of a signal is in general *nonlinear* and very complicated; it depends essentially on context of the signal processing:

$$\phi(s, x) = \phi(s_0, x) + \int_{s_0}^s d\alpha \left(\int_O K(x, y, \alpha; \phi(\alpha, y); \phi(s_0, y)) dy \right), \quad (18)$$

where the kernel K depends on the spatial variables $x, y \in O$ (the distribution of the signal on the brain), on the time variable α , the previous dynamics of the signal $\phi(\alpha, y), \alpha \in [s_0, s)$, and the signal at $s = s_0$, the input. We remark that the dynamics $\phi(s, x)$ depends on input not only additively, i.e., as the initial state which then will evolve in accordance with some integral equation, but even the kernel

of the equation depends nontrivially on the input. Thus the dynamics are different for different inputs.

CSP2. Calculation of correlations; creation of mental images. For each signal $\phi(s, x)$, the brain calculates the corresponding covariance operator $D \equiv D(\phi)$, see (17). The completion of this process, i.e., calculation of D , is associated with creation of the *mental image* MI_ϕ induced by the signal ϕ . Thus on the cognitive level the brain is not interested in the dynamics of the physical signal (18). It is only interested in the dynamics of the covariance operator

$$t \mapsto D(t). \quad (19)$$

We remark that dynamics (18) and (19) have different time scales; the first one is performed on the physical time scale and the second on the mental time scale. Thus it is very important that the “physical brain” and the “cognitive brain” work on two different time scales: the scale of physical signal – τ , and the scale of QLR – T . The interval of time $T \gg \tau$, so its size justifies the ergodic interplay between ensemble and time representations of random signals.

CSP3. Memory of correlations. The density operator D is recorded in the brain. The PCSFT-basis of the model in combination with the ergodic argument make very attractive the following model of memory:

The operator D determines uniquely the Gaussian probability distribution μ_D (with zero mean value). The brain records this probability distribution. How can it do this?

We speculate that, to encode μ_D , the brain uses the statistical distribution by assigning statistical weights to elements of some ensemble Ω .

What are elements of Ω ? They might be neurons or even distributions of chemical components in the brain.

We emphasize once again that such a model of memory for probabilistic laws is based on the ergodicity of processes in the brain: from a signal $\phi(s) \equiv \phi(s, x)$ (the time representation) to its covariance and from the covariance to the probability distribution on an ensemble.⁵

CSP4. Recollection of images. Recollection is the process of activation of a special mental image. We keep to the model of statistical

⁵The choice of a Gaussian probability law can be debated. But at least mathematically it works well, because of the one-to-one correspondence between covariance matrices and Gaussian probability distributions (with zero mean values).

(ensemble) representation of the probability distribution encoding the image, see CSP3. We obtain the following procedure of recollection:

Suppose that a mental image MI_ϕ was recorded in the memory, $\phi \mapsto D = D(\phi) \mapsto \mu = \mu_D$.

The process of recollection: starting with the probability distribution μ the brain computes the covariance operator D of this probability distribution by using the ensemble averaging, see (3), (4).

On the basis of this covariance operator it produces a signal $\phi_{\text{recall}}(s, x)$, $x \in O$, a trajectory of the corresponding Gaussian process.

In this situation, the brain does not reproduce the original signal $\phi(s, x)$, see CSP1. The graphs of $\phi(s, x)$ and $\phi_{\text{recall}}(s, x)$ can differ essentially point wise. Moreover, the original signal $\phi(s, x)$ need not be Gaussian at all. However, correlations inside both signals approximately coincide and, hence, their qualia:

$$\langle f \rangle_\phi = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(\phi(s)) ds \approx \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(\phi_{\text{recall}}(s)) ds = \langle f \rangle_{\phi_{\text{recall}}}.$$

We also remark that even two different recollections $\phi_{\text{recall}}^1(s, x)$ and $\phi_{\text{recall}}^2(s, x)$ of the same image can be very different as physical signals – two different realizations of the same Gaussian process. However, their qualia coincide:

$$\langle f \rangle_{\mu_D} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(\phi_{\text{recall}}^1(s)) ds = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(\phi_{\text{recall}}^2(s)) ds. \quad (20)$$

To be more precise, we say that they coincide approximately, since in reality the brain does not calculate the limit for $T \rightarrow \infty$, but it uses the finite T . Thus

$$\langle f \rangle_{\mu_D} \approx \frac{1}{T} \int_0^T f(\phi_{\text{recall}}^1(s)) ds \approx \frac{1}{T} \int_0^T f(\phi_{\text{recall}}^2(s)) ds. \quad (21)$$

CSP5. Recognition of images. Suppose now that some mental image was saved in the memory: starting with the input signal $\phi(s, x)$ and through its covariance operator $D(\phi)$; for example, the Moscow Kremlin. I came to Moscow once again and I look at the Kremlin; this visual input induces a signal $q(s, x)$. Its covariance operator $D(q)$ is produced, see CSP2. It is compared with covariance operators in the memory to match with the operator $D(\phi)$. (The model under consideration does not describe the mechanism of this comparing process; however, see CSP5n.) Finally, matching of the operators,

$D(\phi) \approx D(q)$, is approached. This activates in the memory the statistical probability distribution $\mu_{D(\phi)}$ in the form of a Gaussian signal $\phi_{\text{recall}}(s, x)$. Its makes the feeling of recognition of the image which was encoded by $D(\phi)$.

6.1 Classical Signal Processing of Mental Images: finite-dimensional approximations

In principle, the brain can calculate the complete covariance operator (17); especially if it works as analogous computational device. However, it consumes a lot of computational resources. We might speculate that the brain selects a finite number of test functions, it is always possible to assume that they are orthogonal in $L_2(O)$:

$$u_1(x), \dots, u_n(x). \quad (22)$$

Instead of the complete covariance operator $D = D(\phi)$, the brain calculates its cutoff, the covariance $n \times n$ matrix $D_n = D_n(\phi)$. Thus, instead of infinite dimensional L_2 -space, the brain works (for a given mental function) in fixed finite dimensional subspace H_n . We modify CSP1-CSP5; the first step CPS1 is not changed. We have:

CSP2n. The signal $\phi(s, x)$ induces the mental image $MI_{\phi;n}$ encoded by the covariance matrix D_n .

CSP3n. The $MI_{\phi;n}$ is recorded in the memory through the probability distribution μ_{D_n} on the finite dimensional Hilbert space H_n .

CSP4n. On the basis of D_n the brain produces a signal $\phi_{\text{recall}}(t)$ in H_n . Its activation is recollection of the memory on $MI_{\phi;n}$.

CSP5n. The memory contains the image $MI_{\phi;n}$ in the form of the matrix $D_n(\phi)$. The new signal produces $MI_{q;n}$ with the covariance matrix $D_n(q)$. These matrices must be compared. Since the whole story is about covariance matrices, so $n \times n$ matrices, it is natural to expect a comparing algorithm which compares cutoffs of these $n \times n$ covariance matrices: first of the dimension two, then three and so on; i.e., first the $D_2(q)$ is compared with 2×2 matrices obtained through projection of mental images on H_2 until the cluster of matrices with the left-up block $D_2(q)(= D_2(\phi))$ is found; then inside this cluster the brain is looking for the sub-cluster with the left-up block $D_3(q)(= D_3(\phi))$ and so on.

How does the brain selects the subspace H_n with the basis (22)?

The most natural is to assume that it just selects a band of frequencies. It is also natural that different mental functions may use different bands, i.e., different Hilbert spaces: for mental functions F and G , two Hilbert spaces H_F and H_G .

7 Quantum-like processing of mental images

We still proceed with functioning of one fixed mental function, say F .

QLP1. Density operator code. At some stage of its growing a cognitive system creates a sufficiently extended database of classical mental images. They are encoded by covariance operators which are transferred in density operators by normalization, see aMI, section 4.0.3. Thus the brain created a collection of QL mental states borrowed from the classical processing, $\rho \in \mathcal{D}_{\text{data}}(H)$ which is a subspace of $\mathcal{D}(H)$. At this stage the brain can be fine by working inside $\mathcal{D}(H)$, i.e., even without contacts with physical and mental environment.

QLP2. Unitary thinking. Processing of information inside $\mathcal{D}(H)$ is the process of QL-thinking. Starting with the operator ρ_0 the brain induces the evolution $\rho(s)$ of the mental QL state. The simplest dynamics corresponds to the process of thinking in the absence of inputs from environment (which includes the body); it is given by the von Neumann equation:

$$i \frac{d\rho(t)}{dt} = [\widehat{\mathcal{H}}, \rho(t)], \quad \rho(0) = \rho_0, \quad (23)$$

where $\widehat{\mathcal{H}} : H \rightarrow H$ is “mental Hamiltonian” (given by a self-adjoint operator). It describes functioning of the mental function F under consideration, cf. [].

In the simplest case Hamiltonian $\widehat{\mathcal{H}}$ is completely determined by the mental function F , so $\widehat{\mathcal{H}} \equiv \widehat{\mathcal{H}}_F$. However, even more complex dynamics seem to be reasonable – with $\widehat{\mathcal{H}}$ which also depends on the initial state $\rho_0 : \widehat{\mathcal{H}} \equiv \widehat{\mathcal{H}}_{F, \rho_0}$, cf. with the QL model of decision making, Chapter ?.

We remark that by starting with e.g. QL-version of a concrete image, i.e., $\rho_0 \in \mathcal{D}_{\text{data}}(H)$, the QL dynamics can go away from this subspace of $\mathcal{D}(H)$. New “really QL” images are created. They can be

visualized through production of corresponding classical signals, see see bMI, section 4.0.3.

We emphasize that the QL-dynamics of mental images is performed on the T -scale which is rough comparing with the τ -scale of physical processing of signals in the brain. Each instant of time t of the T -scale is the (large) interval T of the s -time.

It may be more illustrative to consider the discrete dynamics, the mental time t is considered as the discrete parameter: $t_n = nT$. Then

$$i\rho(t_{n+1}) = T[\widehat{\mathcal{H}}, \rho(t_n)], \rho(0) = \rho_0. \quad (24)$$

QLP3. Dynamics of QL qualia. In QL-processing qualia are reduced to *quadratic functionals* of premental (physical) signals. These functionals are represented by their QL-counterparts – corresponding self-adjoint operators. The evolution of QL qualia is described by the Heisenberg equation:

$$-i\frac{d\widehat{A}(t)}{dt} = [\widehat{\mathcal{H}}, \widehat{A}], \widehat{A}(0) = \widehat{A}_0; \quad (25)$$

or in the discrete representation of the mental time:

$$i\widehat{A}(t_n) = T[\widehat{\mathcal{H}}, \widehat{A}(t_n)], \widehat{A}(0) = \widehat{A}_0. \quad (26)$$

Quale (encoded by \widehat{A}) of a mental image MI_ρ (encoded by the density operator ρ) is quantified by its average given by the quantum formula (2).

Of course, the transition from the class of classical qualia (given by arbitrary functionals of signals) to QL mental features corresponding to only quadratic functionals simplifies mental representation of an image. However, this reduction can be justified by (7) in the framework of QLR (6) of classical functionals of signals.

QLP4. Thinking via operator algebra. This is the crucial point. In QL-thinking the brain switches from the classical physical signal processing, i.e., nonlinear equations of the type (18) to *linear processing* of mental images represented by density operators (26); the representation of qualia is also essentially simplified and it can be done in the linear operator form. Our conjecture is that the brain is really able to realize such linear operator processing of mental entities. This type of processing is especially profitable for “abstract thinking”, i.e., thinking which has a high degree of independence from inputs.

How does the brain realize the QL (operator) processing on the physical level?

We do not know yet. However, we hope that our model may stimulate neurophysiologists to look for the corresponding neuronal representation of QL-processing. We can present the following scheme of mental operator processing:

Since the brain has no other computational resources different from neural electric activity, it seems reasonable to assume that the QL mental dynamics (29), (26) also has to be performed through this activity. The production of density operators can be done similarly to the production of covariance operators in the classical regime. The only difference is that the brain wants to escape the complicated non-linear evolution (18). We consider the following stochastic linear dynamics in Hilbert space H (of classical electromagnetic fields):

$$\frac{\partial \phi}{\partial t}(t, x, \omega) = \widehat{\mathcal{H}}\phi(t, x, \omega), \quad \phi(t_0, x, \omega) = \phi_0(x, \omega), \quad (27)$$

where the random variable $\phi_0(x, \omega)$ is the Gaussian field with zero mean value and the covariance operator ρ_0 . Hence $\langle \rho_0 u, v \rangle = E\langle \phi_0, u \rangle \langle v, \phi_0 \rangle$, $u, v \in H$. The solution of the Cauchy problem (27) is the random field:

$$\phi(t, x, \omega) = U_t \phi_0(x, \omega), \quad (28)$$

where $u_t = e^{-it\widehat{\mathcal{H}}}$ is the standard for QM one parametric group of unitary operators. The covariance operator $\rho(t) \equiv \rho_{\phi(t)}$ can be easily found: $\langle \rho(t)u, v \rangle = E\langle U_t \phi_0, u \rangle \langle v, U_t \phi_0 \rangle = E\langle \phi_0, U_t^* u \rangle \langle U_t^* v, \phi_0 \rangle = \langle \rho(t)U_t^* u, U_t^* v \rangle$. Thus

$$\rho(t) = U_t \rho_0 U_t^*.$$

This operator-valued function $\rho(t)$ satisfies the von Neumann equation (29). Thus the von Neumann evolution of the mental state can be induced by the linear dynamics with random initial condition (27). As was mentioned, the crucial point is that this dynamics is much simpler than the “classical signal dynamics” (18).

Finally, we have the following model of physical realization of the evolution (29). In fact, the brain produces the Gaussian random signal by realizing on the neuronal level the linear Schrödinger type evolution. At each moment of mental time t by calculating its covariance operator the brain creates the mental image given by the covariance-density operator $\rho(t)$.

In quantum information theory it is well known that in general, i.e., in the presence of interaction with environment the von Neumann

equation should be modified to the Gorini-Kossakowski-Sudarshan-Lindblad equation:

$$i\frac{d\rho(t)}{dt} = \widehat{L}(\rho(t)), \quad \rho(0) = \rho_0, \quad (29)$$

where $\widehat{L} : \mathcal{L}(H) \rightarrow \mathcal{L}(H)$ is a linear map with special properties, see []. All previous considerations can be easily generalized to such mental dynamics.

QLP5. Concepts. Consider a subspace L of H and the orthogonal projector $\pi \equiv \pi_{HL} : H \rightarrow L$. It induces the map $\pi : \mathcal{D}(H) \rightarrow \mathcal{D}(L)$,

$$\rho_L \equiv \pi(\rho) = \frac{\pi\rho\pi}{\text{Tr}\pi\rho\pi}.$$

Mental images corresponding to elements of $\mathcal{D}(L)$ can be considered as abstractions of mental images corresponding to elements of $\mathcal{D}(H)$; we call them L -concepts or simply concepts. Take some $\rho_L \in \mathcal{D}(L)$. The mental image MI_{ρ_L} can be interpreted as an abstract concept induced by the cluster of mental images:

$$W_{\rho_L} = \{\rho \in \mathcal{D}(H) : \pi(\rho) = \rho_L\}.$$

Each concept is based on common correlations of a cluster of mental images. It is especially interesting to consider the case $\dim L = m$ and m is quite small. These are very abstract concepts which contains only the basic common correlations in a huge cluster of mental images. It is extremely profitable for the brain to think on the conceptual level; especially to operate in a finite-dimensional L . The operator unitary dynamics (29) is reduced to the matrix dynamics. Conceptual Hamiltonian is given by a symmetric $m \times m$ matrix. For small m , dynamics of such a type are very simple; processing is very rapid.

For example, consider QLR for the concept “house”. In the classical regime the brain created a collection of images of concrete houses MI_1, \dots, MI_k . They were classically encoded by covariance operators D_1, \dots, D_k . These operators contain some common correlations. In the matrix representation they have a common block. For simplicity, suppose that this block is of the diagonal type. Consider a subspace L of H related to this block. Then this block can be represented as a self-adjoint and positive operator D_L in L . We have:

$$D_L = \pi D_j \pi, \quad j = 1, 2, \dots, k,$$

where $\pi : H \rightarrow L$ is the orthogonal projector. Its QL image is given by

$$Q_L = D_L / \text{Tr} D_L = \frac{\pi D_j \pi}{\text{Tr} \pi D_j \pi} = \frac{\pi (D_j / \text{Tr} D_j) \pi}{\text{Tr} \pi (D_j / \text{Tr} D_j) \pi} = \frac{\pi \rho_j \pi}{\text{Tr} \pi \rho_j \pi} = \rho_L.$$

Here ρ_j are QL representations of the covariance operators D_j . The density operator ρ_L gives the abstract concept of house.

We now describe the process of creation of a concept of a higher level of abstractness from a cluster of concepts. Consider a subspace Z of L . The brain can create new concepts belonging to $\mathcal{D}(Z)$ starting with clusters of L -concepts. Of course, it can proceed directly starting with mental images from $\mathcal{D}(H)$. However, such step by step increasing of the level of abstraction is very natural.

QLP6. Neuronal location of the QL-processor. From the general viewpoint there are no reasons to assume that QLR is realized in the same physical domain, the same ensemble of neurons, as classical processing. It may be that there is a special domain O_{QL} which is used for dynamics (29). Our model induced an interesting problem of experimental neurophysiology – to find domains of the brain coupled to QLR. If the hypothesis that the dynamics (29) of mental images is based on the physical dynamics (27) is correct, then domains of QLR can be identified by the presence of Gaussian stochastic dynamics. Unfortunately, at the present level of measurements it is impossible to measure directly the electromagnetic field inside the brain (at least to make measurements in a sufficiently dense set of points). However, even the measurement technology based on EEG provides a possibility of reconstruction of the field inside the brain by using the methods of the inverse problem, [1].

QLP7. Quantum-like consciousness. We may speculate that consciousness can be associated with QL processing in the brain. The von Neumann equation (or more generally the Gorini-Kossakowski-Sudarshan-Lindblad equation) represents the “continuous flow of consciousness”. The feeling of continuity is generated through averaging of physical signals with respect to the mental time scale. i.e., the representation of mental images by covariance-density operators. In fact, on the physical time scale the dynamics is discrete, see (26).

QLP7. Correspondence between classical and quantum qualia. Consider a classical quale given by a functional $f(\phi)$. The corresponding quantum quale is given by the self-adjoint operator, the second derivative of $f(\phi)$ at the point $\phi = 0$. In general behavior

of the functional $f(\phi)$ differs essentially from behavior of its quadratic part. However, on the level of averages the difference is not so large: in the limit $\tau/T \rightarrow 0$ they coincide, see (7).

8 Composite systems

We turn again to physics. In CSM a composite system $S = (S_1, S_2)$ is mathematically described by the Cartesian product of state spaces of its parts S_1 and S_2 . In QM it is described by the tensor product. Majority of researchers working in quantum foundations and, especially quantum information theory, consider this difference in the mathematical representation as crucial. In particular, entanglement which is a consequence of the tensor space representation is considered as totally nonclassical phenomenon. However, we recall that Einstein considered the EPR-states as exhibitions of classical correlations due to the common preparation. PCSFT will realize Einstein's dream on entanglement.

Let $S = (S_1, S_2)$, where S_i has the state space H_i – complex Hilbert space. Then by CSM the state space of S is $H_1 \times H_2$. By extending PCSFT to composite systems we should describe ensembles of composite systems by probability distributions on this Cartesian product, or by a random field $\phi(x, \omega) = (\phi_1(x, \omega), \phi_2(x, \omega)) \in H_1 \times H_2$.

In our approach each quantum system is described by its own random field: S_i by $\phi_i(x, \omega)$, $i = 1, 2$. However, these fields are CORRELATED – in completely classical sense. Correlation at the initial instant of time $s = s_0$ propagates in time in the complete accordance with laws of QM. There is no action at the distance. It is a purely classical dynamics of two stochastic processes which were correlated at the beginning. (In fact, the situation is more complex: there is also the common random background, vacuum fluctuations; we shall come back to this question a little bit later).

8.1 Operator realization of wave function

Consider now the QM-model, take a pure state case: $\Psi \in H_1 \otimes H_2$. Can one peacefully connect the QM and PCSFT formalisms? Yes! But Ψ should be interpreted in completely different way than in the conventional QM.

The main mathematical point: Ψ is not vector! It is an operator!

It is, in fact, the non-diagonal block of the covariance operator of the corresponding prequantum random field: $\phi(x, \omega) \in H_1 \times H_2$. The wave function $\Psi(x, y)$ of a composite system determines the integral operator:

$$\widehat{\Psi}\phi(x) = \int \Psi(x, y)\phi(y)dy.$$

We keep now to the finite-dimensional case. Any vector $\Psi \in H_1 \otimes H_2$ can be represented in the form $\Psi = \sum_{j=1}^m \psi_j \otimes \chi_j$, $\psi_j \in H_1, \chi_j \in H_2$, and it determines a linear operator from H_2 to H_1

$$\widehat{\Psi}\phi = \sum_{j=1}^m (\phi, \chi_j)\psi_j, \quad \phi \in H_2. \quad (30)$$

Its adjoint operator $\widehat{\Psi}^*$ acts from H_1 to H_2 : $\widehat{\Psi}^*\psi = \sum_{j=1}^m (\psi, \psi_j)\chi_j$, $\psi \in H_1$. Of course, $\widehat{\Psi}\widehat{\Psi}^* : H_1 \rightarrow H_1$ and $\widehat{\Psi}^*\widehat{\Psi} : H_2 \rightarrow H_2$ and these operators are self-adjoint and positively defined. Consider the density operator corresponding to a pure quantum state, $\rho = \Psi \otimes \Psi$. Then the operators of the partial traces $\rho^{(1)} \equiv \text{Tr}_{H_2}\rho = \widehat{\Psi}\widehat{\Psi}^*$ and $\rho^{(2)} \equiv \text{Tr}_{H_1}\rho = \widehat{\Psi}^*\widehat{\Psi}$.

8.2 Basic equality

Let $\Psi \in H_1 \otimes H_2$ be normalized by 1. Then, for any pair of linear bounded operators $\widehat{A}_j : H_j \rightarrow H_j, j = 1, 2$, we have:

$$\text{Tr}\widehat{\Psi}\widehat{A}_2\widehat{\Psi}^*\widehat{A}_1 = \langle \widehat{A}_1 \otimes \widehat{A}_2 \rangle_{\Psi} \equiv (\widehat{A}_1 \otimes \widehat{A}_2 \Psi, \Psi). \quad (31)$$

This is a mathematical theorem[33]; it will play a fundamental role in further considerations.

8.3 Coupling of classical and quantum correlations

In PCSFT a composite system $S = (S_1, S_2)$ is mathematically represented by the random field $\phi(\omega) = (\phi_1(\omega), \phi_2(\omega)) \in H_1 \times H_2$. Its covariance operator D has the block structure

$$D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix},$$

where $D_{ii} : H_i \rightarrow H_i, D_{ij} : H_j \rightarrow H_i$. The covariance operator is self-adjoint. Hence $D_{ii}^* = D_{ii}$, and $D_{12}^* = D_{21}$.

Here by the definition: $(D_{ij}u_j, v_i) = E(u_j, \phi_j(\omega))(v_i, \phi_i(\omega)), u_i \in H_i, v_j \in H_j$. For any Gaussian random vector $\phi(\omega) = (\phi_1(\omega), \phi_2(\omega))$ having zero average and any pair of operators $\hat{A}_i \in \mathcal{L}_s(H_i), i = 1, 2$, the following equality takes place: $\langle f_{A_1}, f_{A_2} \rangle_\phi \equiv Ef_{A_1}(\phi_1(\omega))f_{A_2}(\phi_2(\omega)) = (\text{Tr}D_{11}\hat{A}_1)(\text{Tr}D_{22}\hat{A}_2) + \text{Tr}D_{12}\hat{A}_2D_{21}\hat{A}_1$. We remark that $\text{Tr}D_{ii}\hat{A}_i = Ef_{A_i}(\phi_i(\omega)), i = 1, 2$. Thus we have $f_{A_1}f_{A_2} = Ef_{A_1}Ef_{A_2} + \text{Tr}D_{12}\hat{A}_2D_{21}\hat{A}_1$. Consider a Gaussian vector random field such that $D_{12} = \hat{\Psi}$:

$$E(f_{A_1} - Ef_{A_1})(f_{A_2} - Ef_{A_2}) = (\hat{A}_1 \otimes \hat{A}_2 \hat{\Psi}, \Psi) \equiv \langle \hat{A}_1 \otimes \hat{A}_2 \rangle_\Psi, \quad (32)$$

or, for covariance of two classical random vectors f_{A_1}, f_{A_2} , we have: $\text{cov}(f_{A_1}, f_{A_2}) = \langle \hat{A}_1 \otimes \hat{A}_2 \rangle_\Psi$.

We have the following equality for averages of quadratic forms of coordinates of the prequantum random field describing the state of a composite system: $Ef_{A_i}(\phi_i(\omega)) = \text{Tr}D_{ii}\hat{A}_i$. We want to construct a random field such that these averages will match those given by QM. For the latter, we have: $\langle \hat{A}_1 \rangle_\Psi = (\hat{A}_1 \otimes I_2 \Psi, \Psi) = \text{Tr}(\Psi \Psi^* \hat{A}_1); \langle \hat{A}_2 \rangle_\Psi = (I_1 \otimes \hat{A}_2 \Psi, \Psi) = \text{Tr}(\hat{\Psi}^* \hat{\Psi} \hat{A}_2)$, where I_i denotes the unit operator in $H_i, i = 1, 2$. Thus it would be natural to take $D_\Psi = \begin{pmatrix} \hat{\Psi} \hat{\Psi}^* & \hat{\Psi} \\ \hat{\Psi}^* & \hat{\Psi}^* \hat{\Psi} \end{pmatrix}$.

However, *this operator is not positively defined!* It could not determine any probability distribution on the space of classical fields. We modify it to obtain a positively defined operator. Originally this modification had purely mathematical reasons, but there are deep physical grounds for it.

The operator $\tilde{D}_\Psi = \begin{pmatrix} \hat{\Psi} \hat{\Psi}^* + \epsilon I & \hat{\Psi} \\ \hat{\Psi}^* & \hat{\Psi}^* \hat{\Psi} + \epsilon I \end{pmatrix}$ is positively defined if $\epsilon > 0$ is large enough [33]. Hence, it determines uniquely the Gaussian measure on the space of classical fields. Suppose now that $\phi(\omega)$ is a random vector with the covariance operator \tilde{D}_Ψ . Then

$$\langle \hat{A}_1 \rangle_\Psi = Ef_{A_1}(\phi_1(\omega)) - \epsilon \text{Tr} \hat{A}_1. \quad (33)$$

This relation for averages and relation (32) provide coupling between PCSFT and QM. Quantum statistical quantities can be obtained from corresponding quantities for classical random field: *“irreducible quantum randomness” is reduced to randomness of classical prequantum fields.*

8.4 Vacuum fluctuations

The additional term given by the unit operator in the diagonal blocks of the covariance operator of the prequantum vector field corresponds to the field of the white noise type. Such a field can be considered as vacuum fluctuations, vacuum field. PCSFT induces the following picture of reality:

Fluctuations of the vacuum field are combined with random fields representing quantum systems. Since we cannot separate, e.g., electron from the vacuum field, we cannot separate totally any two quantum systems. Thus all quantum systems are “entangled” via the vacuum field.

8.5 Superstrong quantum correlations

In PCSFT such correlations (violating Bell’s inequality) are due to the presence of the vacuum field. The off-diagonal term $\hat{\Psi}$ can be so large only if the diagonal terms are completed by the contribution of the vacuum field. Mathematics tells us this. Thus they are so strong, because the vacuum field really couple any two systems; they are in the same fluctuating space.

Space is a huge random wave; quantum systems are spikes on this wave; they are correlated via this space-wave. Thus quantum correlations have two contributions:

- 1) initial preparation;
- 2) coupling via the vacuum field.

The picture is pure classical... In this model the vacuum field is the source of additional correlations. It seems that this classical vacuum field is an additional (purely classical) quantum computational resource.

9 Entanglement of mental functions

Consider now two mental functions F_1 and F_2 which are spatially coupled to domains O_1 and O_2 of the brain.

9.1 Classical correlated processing of mental images

Consider the input signal ϕ_0 which is split into sub-signals used by F_1 and F_2 , respectively: $\phi_0 = (\phi_{01}, \phi_{02})$. We emphasize that these are correlated signals; i.e., in general the nondiagonal term of their covariance matrix D_0 is nonzero. Starting with ϕ_0 the F_1 and F_2 produce the random field $\phi(t, x, \omega) = (\phi_1(t, x, \omega), \phi_2(t, x, \omega))$. In the general case F_1 and F_2 can interact on the physical level. But in principle they can proceed their signals $\phi_1(t, x, \omega)$ and $\phi_2(t, x, \omega)$ without any physical interaction (with negligibly small interaction). For example, domains O_1 and O_2 are sufficiently separated in space. We shall concentrate our study on such a case, since here the QL features of the model are the most visible.

Thus the brain produces the covariance operator $D(t) = D_{\phi(t)}$ representing the classical mental image at the moment t of mental time. Its dynamics is very complicated, it is induced by in general nonlinear dynamics of corresponding signals $\phi_1(t, x, \omega)$ and $\phi_2(t, x, \omega)$.

We now coming to an important issue of our model, namely, the *role of noise in information processing*. The presence of noise in the brain can be considered as a disturbing effect for deterministic information processing. However, in our model mental images are encoded by covariance operators. Incorporation of the noisy contribution in the covariance operator does not induce problems. Moreover, it can improve (!) information processing by coupling through noise spatially separated processes in the brain.

Let us consider noise of the white noise type, i.e., the Gaussian random field with zero mean value and the covariance operator $D_{\text{noise}} = \epsilon I, \epsilon > 0$. We now suppose that this noise is incorporated into the signal $\phi(t, x, \omega)$ produced by (F_1, F_2) . Hence,

$$D(t) = \tilde{D}(t) + D_{\text{noise}}.$$

We remark that nondiagonal blocks describing correlations between mental functions F_1 and F_2 do not depend on the presence of the noise. However, the presence of the noise provides a possibility to make these nondiagonal blocks essentially larger (in the operator sense) than in its absence. Thus the noise coordinates processes in O_1 and O_2 : it produces so to say nonlocal correlations.

9.2 Quantum-like entangled processing of mental images

We now can repeat the QL-story on the information processing by considering mental images encoded by density operators of the class $\mathcal{D}(H_1 \otimes H_2)$. It is possible to embed this space into the space of all covariance operators on the Cartesian product $H_1 \times H_2$. We have done this for the density operators corresponding to pure quantum states $\Psi \in H_1 \otimes H_2$, see section 8.3. It is possible to generalize our construction to arbitrary density operators in $H_1 \otimes H_2$, see []. The dynamics of the mental state is described by von Neumann equation (or more generally the Gorini-Kossakowski-Sudarshan-Lindblad equation) in $\mathcal{D}(H_1 \otimes H_2)$. Each pair of qualia \hat{A}_1 and \hat{A}_2 associated with the mental functions F_1 and F_2 forms a new quale which is given by the operator $\hat{A}_1 \otimes \hat{A}_2$. It is quantified by its average.

The neuronal realization of the QL-dynamics can be performed in the same way as it was done for a single mental function, see QLP4. Consider the case of two isolated processing: in O_1 and O_2 . Each of them is described by mental Hamiltonian $\hat{\mathcal{H}}_i : H \rightarrow H, i = 1, 2$. In this case QM dynamics is given by the Schrödinger equation with Hamiltonian $\hat{\mathcal{H}} = \hat{\mathcal{H}}_1 \otimes I + I \otimes \hat{\mathcal{H}}_2$:

$$\frac{\partial \psi}{\partial t}(t, x_1, x_2) = \hat{\mathcal{H}}\psi(t, x_1, x_2), \quad \psi(t_0, x_1, x_2) = \psi_0(x_1, x_2). \quad (34)$$

We remark that, since in general the initial wave function is not factorizable, i.e., $\psi(x_1, x_2) \neq \psi_{01}(x_1)\psi_{02}(x_2)$, the Schrödinger equation cannot be split into a system of equations corresponding to the subsystems of the composite system. However, we can do this on the level of the prequantum stochastic process and obtain a system of linear equations with random initial conditions, $j = 1, 2$:

$$\frac{\partial \phi}{\partial t}(t, x_j, \omega) = \hat{\mathcal{H}}_j \phi(t, x_j, \omega), \quad \phi(t_0, x_j, \omega) = \psi_0(x_j, \omega). \quad (35)$$

The covariance operator $\rho(t) = \rho_{(\phi_1(t), \phi_2(t))}$ induced by the solution of this system is the solution of the von Neumann equation for the composite system.

Correlations between processes related to F_1 and F_2 are suppre-strong! They can violate Bell's inequality! However, in our model this is not a consequence of mystical quantum nonlocality, but of the presence of the white noise which increases the correlation effect.

In “prequantum physics” this noise corresponds to the background (zero point field), it represents vacuum fluctuations. In the brain this is the ordinary macroscopic noise of a huge electric network consisting of billions of neurons.

References

- [1] Hameroff, S.: Quantum coherence in microtubules. A neural basis for emergent consciousness? *J. of Cons. Stud.* 1, 91-118 (1994)
- [2] Hameroff, S.: Quantum computing in brain microtubules? The Penrose-Hameroff Orch Or model of consciousness. *Phil. Tr. Royal Sc., London A* 1-28, (1994)
- [3] R. Penrose, *The emperor’s new mind*. Oxford Univ. Press, New-York (1989)
- [4] Penrose, R.: *Shadows of the Mind*. Oxford University Press, Oxford (1994)
- [5] Vitiello, G.: *My Double Unveiled - the Dissipative Quantum Model of Brain*. J. Benjamins Publ. Company, Amsterdam/Philadelphia (2001)
- [6] D. Aerts and S. Aerts, Applications of quantum statistics in psychological studies of decision-proceses. *Foundations of Science* 1, 1-12 (1995).
- [7] J. A. Barrett, *The quantum mechanics of minds and worlds*. Oxford Univ. Press, Oxford, 1999.
- [8] A. Yu. Khrennikov, *Classical and quantum mental models and Freud’s theory of unconscious mind*. Växjö Univ. Press, Växjö, 2002.
- [9] A. Yu. Khrennikov, *Information Dynamics in Cognitive, Psychological and Anomalous Phenomena*. Kluwer Academic, Dordrecht, 2004.
- [10] A. Khrennikov, *Open Systems and Information Dynamics* 11 (3), 267-275 (2004).
- [11] A. Yu. Khrennikov, Quantum-like brain: Interference of minds. *BioSystems* 84, 225–241 (2006).
- [12] E. Conte, O. Todarello, A. Federici, F. Vitiello, M. Lopane, A. Khrennikov, and J.P: Zbilut, J Some remarks on an experiment

- suggesting quantum-like behavior of cognitive entities and formulation of an abstract quantum mechanical formalism to describe cognitive entity and its dynamics. *Chaos, Solitons and Fractals*, **31**, 1076-1088 (2006).
- [13] Busemeyer, J. B., Wang, Z. and Townsend, J. T.: Quantum dynamics of human decision making. *J. Math. Psychology* 50, 220-241 (2006)
- [14] Busemeyer, J. R., Matthews, M., and Wang, Z.: A Quantum Information Processing Explanation of Disjunction Effects. In: Sun, R. and Myake, N. (eds.) *The 29th Annual Conference of the Cognitive Science Society and the 5th International Conference of Cognitive Science* (Pp. 131-135) Mahwah, NJ. Erlbaum (2006)
- [15] Busemeyer, J. R., Santuy, E., Lambert-Mogiliansky, A.: Comparison of Markov and quantum models of decision making. In P. Bruza, W. Lawless, K. van Rijsbergen, D. A. Sofge, B. Coeke, S. Clark (Eds.) *Quantum interaction: Proceedings of the Second Quantum Interaction Symposium*, pp.68-74. London: College Publications, (2008)
- [16] E. Conte, A. Yu. Khrennikov, O. Todarello, A. Federici, J. P. Zbilut, On the existence of quantum wave function and quantum interference effects in mental states: an experimental confirmation during perception and cognition in humans, *Neuroquantology* **7**, N 2 (2009).
- [17] E. Conte, A. Khrennikov, O. Todarello, A. Federici, J. P. Zbilut, Mental States Follow Quantum Mechanics during Perception and Cognition of Ambiguous Figures. *Open Systems and Information Dynamics*, **16**, 1-17 (2009).
- [18] L. Accardi, A. Khrennikov, M. Ohya, Quantum Markov Model for Data from. Shafir-Tversky Experiments in Cognitive Psychology. *Open Systems and Information Dynamics*, **16**, 371385 (2009).
- [19] K.-H.Fichtner, L.Fichtner, W.Freudenberg and M.Ohya, On a quantum model of the recognition process. *QP-PQ:Quantum Prob. White Noise Analysis* **21**, 64-84 (2008).
- [20] E. Haven and A. Khrennikov, Quantum mechanics and violation of the sure-thing principle: the use of probability interference and other concepts. *J. Math. Psychology*, **53**, 378-388 (2009).
- [21] A. Khrennikov, Quantum-like model of cognitive decision making and information processing. *Biosystems*, **95**, 179-187 (2009).

- [22] A. Khrennikov, *Ubiquitous quantum structure: from psychology to finance*, Springer, Heidelberg- Berlin-New York, 2010.
- [23] A. Khrennikov, *Interpretations of Probability*. De Gruyter, Berlin (2009)
- [24] A. Khrennikov, *Contextual Approach to Quantum Formalism*, Springer, Berlin (2009).
- [25] Kolmogoroff, A. N.: *Grundbegriffe der Wahrscheinlichkeitsrechnung*. Springer Verlag, Berlin (1933); English translation: Kolmogorov, A.N.: *Foundations of the Probability Theory*. Chelsea Publishing Company, New York (1956)
- [26] Shafir, E. and Tversky, A.: Thinking through uncertainty: non-consequential reasoning and choice. *Cognitive Psychology* 24, 449-474 (1992)
- [27] Tversky, A. and Shafir, E.: The disjunction effect in choice under uncertainty. *Psychological Science*, 3, 305-309 (1992)
- [28] Savage, L.J.: *The foundations of statistics*. New York: Wiley and Sons (1954).
- [29] M. Asano, A. Khrennikov, and M. Ohya, Quantum-like Model for Decision Making Process in Two Players Game. To be published in *Foundations of Physics* (2011)
- [30] B. Schack, N. Vath, H. Petsche, H. -G. Geissler, E. Mller, *Int. J. Psychophysiology* 44, 143-163 (2002).
- [31] A. Khrennikov, *J. Phys. A: Math. Gen.* **38**, 9051-9073, 2005.
- [32] A. Khrennikov, *Physics Letters A*, **372**, 6588-6592 (2008).
- [33] A. Khrennikov, *Europhysics Letters* **88**, 40005 (2009).
- [34] H. F. Hofmann, Quantum noise and spontaneous emission in semiconductor lasers devices. PhD-thesis, Studgard, 1999.
- [35] A. Khrennikov, *J. Consciousness Studies* **15**, 10-25 (2009).
- [36] 't Hooft, G.: Quantum mechanics and determinism. hep-th/0105105 (2001)
- [37] 't Hooft, G.: The free-will postulate in quantum mechanics. quant-ph/0701097 (2007)
- [38] A. Einstein and L. Infeld, *Evolution of Physics: The Growth of Ideas from Early Concepts to Relativity and Quanta*, Simon and Schuster, 1961.

- [39] D. Amit, *Modeling Brain Function: The World of Attractor Neural Networks*. Cambridge Uni. Press, Cambridge, 1989.
- [40] A. Khrennikov, Quantum randomness as a result of random fluctuations at the Planck time scale? *Int. J. Theor. Phys.* **47** 114-124 (2008).